Lesson 21: Rational Functions

Definition 1. Rational Function: *A function f that can be expressed in the form*

$$f(x) = \frac{N(x)}{D(x)},$$

*where the numerator $N(x)$ and the denominator $D(x)$ are polynomials and $D(x)$ is not the zero polynomial, is called a **rational function**. The domain of $f(x)$ is all real numbers for which $D(x) \neq 0$.*

Example 1. *The function $f(x) = \frac{x^2-2x+5}{x^2-2x-3}$ is a rational function with domain equals to $(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$.*

Definition 2. Vertical Asymptote: *The **line** $x = a$ is called a **vertical asymptote** of the graph of a function $f(x)$ **if** $|f(x)| \to \infty$ as $x \to a^+$ or $x \to a^-$.*

Definition 3. Locating Vertical Asymptotes of Rational Functions: *If $f(x) = \frac{N(x)}{D(x)}$ is a rational function, where $N(x)$ and $D(x)$ **DO NOT have a common factor** and a is a real zero of $D(x)$, then the **line** with equation $x = a$ is a **vertical asymptote** of the graph of $f(x)$.*

Example 2. *Find the vertical asymptote of $f(x) = \frac{x+2}{x^2-4}$.*

Solution: *To locate the vertical asymptotes of $f(x)$, we need to factor both the numerator and the denominator:*

$$\begin{aligned} f(x) &= \frac{x+2}{x^2-4} \\ &= \frac{x+2}{(x+2)(x-2)} \\ &= \frac{1}{x-2}. \quad \textit{Cancel out } (x+2) \end{aligned}$$

*Since the numerator $N(x) = 1$ and $D(x) = x - 2$ has no common factor, then solving the equation $x - 2 = 0$ implies that $x = 2$ is a vertical asymptote of $f(x)$. **Note** that since $x = -2$ is not in the domain, then we were able to cancel out the factor $(x + 2)$ in $f(x)$.*

Definition 4. Horizontal Asymptote: *The **line** $y = k$ is called a **horizontal asymptote** of the graph of a function $f(x)$ **if** $f(x) \to k$ as $x \to \infty$ or as $x \to -\infty$.*

Definition 5. Locating Horizontal Asymptotes of Rational Functions: *Let $f(x) = \frac{N(x)}{D(x)}$ be a rational function. To locate the horizontal asymptote of*

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0},$$

*we compare the **degrees** n and m:*

1. *If $n < m$, then the x-axis or $y = 0$ is the horizontal asymptote.*

2. *If $n = m$, then the line $y = \frac{a_n}{b_m}$ is the horizontal asymptote.*

*3. If $n > m$, then the the graph of $f(x)$ has **NO** horizontal asymptote.*

Example 3. *Find the horizontal asymptotes of $f(x) = \frac{x+3}{x^2-9}$.*

Solution: *To find the horizontal asymptote, we compare the degree of the numerator with the degree of the denominator, Since $n = 1$ and $m = 2$. So $n < m$ which means that $y = 0$ is the horizontal asymptote.*

Definition 6. **x-intercepts of Rational Functions:** *To find the x-intercepts of a rational function $f(x) = \frac{N(x)}{D(x)}$, then write $f(x)$ in lowest terms and solve the equation $N(x) = 0$.*

Example 4. *Find the x-intercepts of $f(x) = \frac{x+3}{x^2-4}$.*

Solution: *To find the x-intercepts of $f(x) = \frac{x+3}{x^2-4}$, solve the equation $x + 3 = 0$, which implies $x = -3$. Then $(-3, 0)$ is the x-intercept.*

Definition 7. **y-intercepts of Rational Functions:** *To find the y-intercepts of a rational function $f(x) = \frac{N(x)}{D(x)}$, then put $x = 0$ and find $f(0)$.*

Example 5. *Find the y-intercepts of $f(x) = \frac{x+3}{x^2-4}$.*

Solution: *To find the y-intercepts of $f(x) = \frac{x+3}{x^2-4}$, find $f(0)$.*

$$\begin{aligned} f(0) &= \frac{0+3}{(0)^2-4} \\ &= \frac{3}{-4} = -\frac{3}{4}. \end{aligned}$$

Then the y-intercept is $(0, -\frac{3}{4})$.

Example 6. *Find a rational function $f(x) = \frac{N(x)}{D(x)}$ that has a vertical asymptote at $x = 2$, a horizontal asymptote at $y = 1$ and a y-intercept at $(0, -2)$.*

Solution: *Since $f(x)$ has a horizontal asymptote at $y = 1$, then **the degrees** of $N(x)$ and $D(x)$ are the **same** and the **coefficients** a_n **and** b_m **are equal** as well. Then we can write $f(x) = \frac{x+c}{x+d}$.*

*To find vertical asymptotes, **solve** $x + d = 0$ which implies that $x = -d$ and $x = 2$ is given to be the vertical asymptote of $f(x)$. Then $d = -2$. Hence $f(x) = \frac{x+c}{x+d} = \frac{x+c}{x-2}$.*

Since $(0, -2)$ is the y-intercept, then $-2 = \frac{0+c}{0-2} = \frac{c}{-2}$. Hence $c = 4$. Therefore, $f(x) = \frac{x+4}{x-2}$.